Today, finish 14.7 and then do 15.1

<u>14.7 (continued): Global Max/Min</u>

2D Intro to Global Max/Min

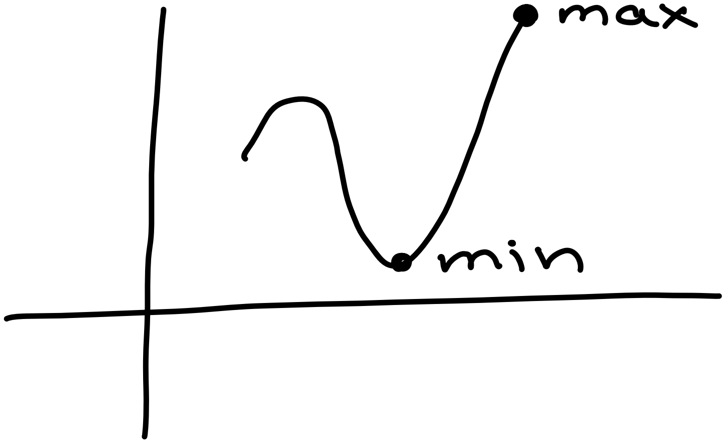
First, a review of calculus 1…

@ critical or end points

Do you remember how to do this?

Find the global max/min of:

A) $f(x) = 2x^2 + 9$ on $0 \leq x \leq 2$.
B) $g(x) = 4x^2 - 4x + 9$ on $0 \leq x \leq 2$.
C) $h(y) = 9$ on $0 \leq y \leq 4$.



$\boxed{A}$ $z = 2x^2 + 9$ $0 \leq x \leq 2$

$z' = 4x = 0 \rightarrow x = 0$

$x = 0 \rightarrow z = \textcircled{9}$ lowest

$x = 2 \rightarrow z = 2(4) + 9 = \textcircled{17}$ highest

$\boxed{B}$ $z = 4x^2 - 4x + 9$ $0 \leq x \leq 2$

$z' = 8x - 4 = 0 \rightarrow x = 1/2 \rightarrow z = 1 - 2 + 9 = \textcircled{8}$ min

$x = 0 \rightarrow z = 9$

$x = 2 \rightarrow z = \textcircled{17}$ max

$\boxed{C}$ $z = 9$ $0 \leq y \leq 4$

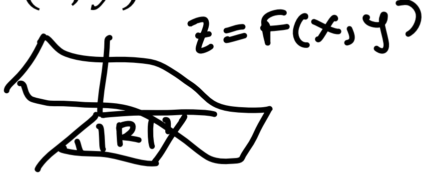
$z' = 0 = 0 \rightarrow$ everything is a critical #

$z = \textcircled{9} \leftarrow$ max + min

Global (or Absolute) Max/Min:

Given

- A surface: $z = f(x, y)$
- A region on the xy-plane: R



Want

Biggest and smallest z over this region.

How

Step 1: Draw Region, label sides
Step 2: "Inside". Find all critical pts.
Step 3: "Boundary"

Over each boundary curve, plug into surface and solve the resulting one variable max/min problem.

Keep track of your outputs throughout.

biggest z = global max
smallest z = global min

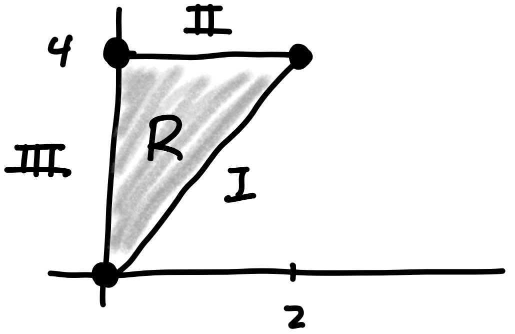
We will do an example on the next slide, but also see online for a video with an additional example

F'18 – Exam 2 – Loveless

Find the global max of → biggest output on range

$$f(x,y) = 4x^2 - xy + 9$$

over the triangular region with corners at (0,0), (0,4), and (2,4).



"Inside":

$f_x = 8x - y = 0$

$f_y = -x = 0 \rightarrow x = 0 \xrightarrow{①} 8(0) - y = 0 \rightarrow y = 0 \quad (0,0)$

"Boundary": each side of triangle

$\boxed{I}$ $y = 2x \longrightarrow z = 4x^2 - x(2x) + 9 \rightarrow z = 2x^2 + 9 \quad 0 \le x \le 2$

$z = 9 \quad \boxed{z = 17}$ max

$\boxed{II}$ $y = 4 \rightarrow z = 4x^2 - 4x + 9 \quad 0 \le x \le 2$

$z = 9 \quad \underset{\text{min}}{\boxed{z = 8}} \quad \underline{z = 17}$ → same point

worked out above!

$\boxed{III}$ $x = 0 \rightarrow z = 9 \quad 0 \le y \le 4$

$z = 9$

- Solutions (see 3(b))
- *Visuals: https://www.math3d.org/5lTfUKmA*

15.1 Iterated Integrals – Double Integral Intro

Try this…

- Treat y as a constant.
- Integrate "inside" with respect to x.
- Then integrate with respect to y.

$z = f(x,y) = 8xy^2$

$$\int_0^2 \left(\int_1^2 8xy^2 \, dx \right) dy =$$

$\int_0^2 \left(8\frac{1}{2}x^2y^2 \Big|_1^2 \right) dy$

$\int_0^2 \left(4(2)^2y^2 - 4(1)^2y^2 \right) dy$

$\int_0^2 12y^2 \, dy$ cross sectional area @ y

$4y^3 \Big|_0^2$

$\boxed{32}$

same answer

Try again in the reversed order:

$$\int_1^2 \left(\int_0^2 8xy^2 \, dy \right) dx =$$

$\int_1^2 \left(8x\frac{1}{3}y^3 \Big|_0^2 \right) dx$

$\int_1^2 \frac{64}{3}x \, dx$ cross sectional area @ x

$\frac{32}{3}x^2 \Big|_1^2$

$\boxed{32}$

Visuals: https://www.math3d.org/XZgRT8IT

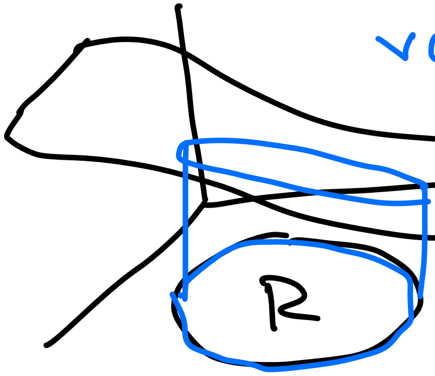
15.1-15.2 Theory Overview

Given

1. $z = f(x,y)$
2. A region, *R*, in the *xy*-plane.

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

= `signed' volume between *f(x,y)* and *R*.



volume between region and surface

General Notes:

Gives a number!

- If *f(x,y)* is above the *xy*-plane it is *positive*.
- If *f(x,y)* is below the *xy*-plane it is *negative*.

Symbol Notes:

ΔA = area of base = $\Delta x\Delta y = \Delta y\Delta x$

$f(x_{ij},y_{ij})\Delta A$ = (height)(area of base)

= volume of one approximating box

Units of $\iint_R f(x,y)dA$ are

(units of f(x,y))(units of x)(units of y)

Theory/Approximation Example

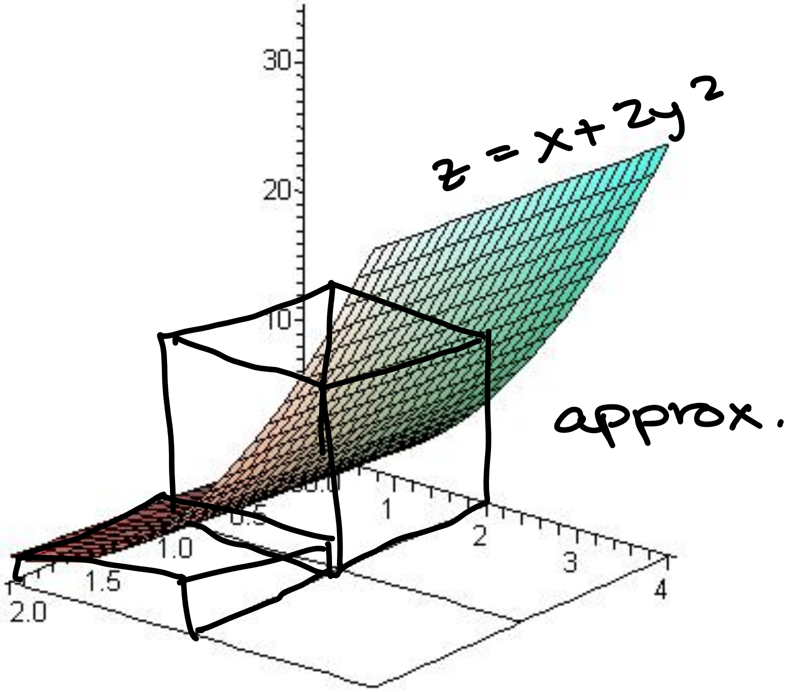
Example: Estimate the volume under

$$z = f(x,y) = x + 2y^2$$

and above

R = [0,2] x [0,4]

$= \{(x,y) : \underbrace{0 \le x \le 2}, \underbrace{0 \le y \le 4}\}$

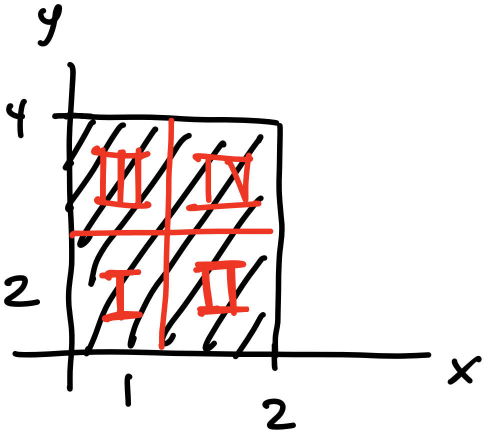


(a) Break the region R into m = 2 columns and n = 2 rows; 4 sub-regions;

(b) Approx. using a rectangular box over each region (use *upper-right* endpts).

$$\iint_R x+2y^2 \, dA \approx (\text{Height})(\text{Area of Base})$$

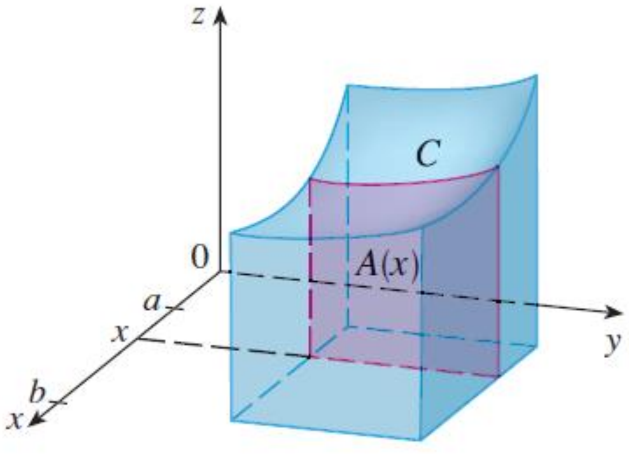
$$\approx \overset{(1+2(2)^2)\ \text{I}}{f(1,2)\Delta A} + \overset{(2+2(2)^2)\ \text{II}}{f(2,2)\Delta A} \ \ldots \text{etc.}$$



Iterated Integrals Theory

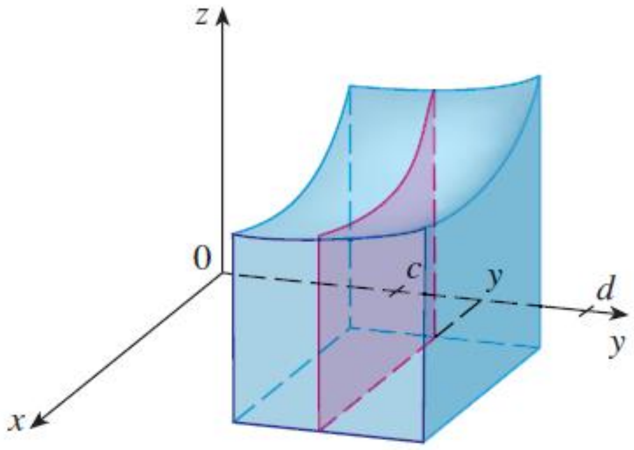
If you fix x: The area under this curve is

$$\int_c^d f(x,y)dy = \text{"cross sectional area under the surface at this fixed } x \text{ value"}$$



If you fix y: The area under this curve

$$\int_a^b f(x,y)dx = \text{"cross sectional area under the surface at this fixed } y \text{ value"}$$



From Math 125,

$$\text{Vol} = \int_a^b \text{Area(x)}dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

$$\text{Vol} = \int_c^d \text{Area}(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Two basic examples to help you visualize:

Visuals for (a): https://www.math3d.org/4HzHYflJ

Visuals for (b): https://www.math3d.org/FL8kzBB1

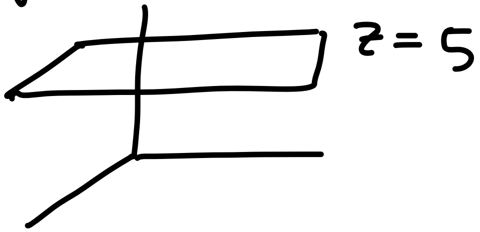
You try to evaluate

$$(a)\ \int_0^3\left(\int_0^2 5\,dx\right)dy$$

z=5 ← plane 5 units up

area of a slice

over region



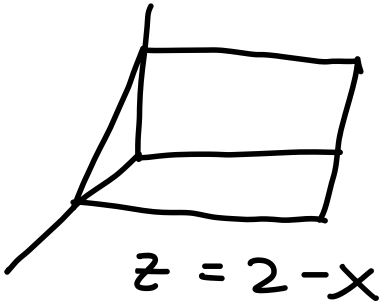
$$\int_0^3 (5x|_0^2)\,dy$$

$$\int_0^3 10\,dy \Rightarrow 10y|_0^3 = \boxed{30 \text{ ft}^3}$$

$$(b)\ \int_0^3\left(\int_0^2 2-x\,dx\right)dy$$

z=2-x

$$4-2=2$$



$$\int_0^3 (2x-\tfrac{1}{2}x^2)\Big|_0^2\,dy$$

$$\int_0^3 2\,dy \Rightarrow 2y|_0^3 = \boxed{6 \text{ ft}^3}$$

$$0\le x\le 2 \qquad 0\le y\le 3$$

“Hard” 15.1 HW Problem

Compute

$$\iint_R \frac{2x}{1+xy}\,dA$$

over $R = [0,2]\ x\ [0,1]$.

Visuals: https://www.math3d.org/Ig0u6x7d

Preview of 15.2

Evaluate

$$\int_0^7 \int_{y/7}^1 2e^x \, dx \, dy$$

Also try evaluating (this should come out to the same number, see if it does)

$$\int_0^1 \int_0^{7x} 2e^x \, dy \, dx$$

Visual: https://www.math3d.org/YKdmMST9

Examples (like 15.1 HW):

1. Find the volume under
   $z = x + 2y^2$ and
   above $0 \leq x \leq 2, \quad 0 \leq y \leq 4$

I will not do all these in class, but you can try them out for more practice and see full solutions in my old lecture notes.

2. $\int_0^3 \int_0^1 2xy\sqrt{x^2 + y^2}dxdy$

3. Find the double integral of

$$f(x, y) = y\cos(x + y)$$

over the rectangular region

$$0 \leq x \leq \pi, \ \ 0 \leq y \leq \pi/2$$

“Hard” 15.1 HW Problem

Compute

$$\iint\limits_{R} \frac{2x}{1+xy}\, dA$$

over $R = [0,2]\ x\ [0,1]$.

Visuals: https://www.math3d.org/Ig0u6x7d

Intro to 15.2

Evaluate

$$\int_0^7 \int_{y/7}^1 2e^x \, dx \, dy$$

Also try evaluating (this should come out to the same number, see if it does)

$$\int_0^1 \int_0^{7x} 2e^x \, dy \, dx$$

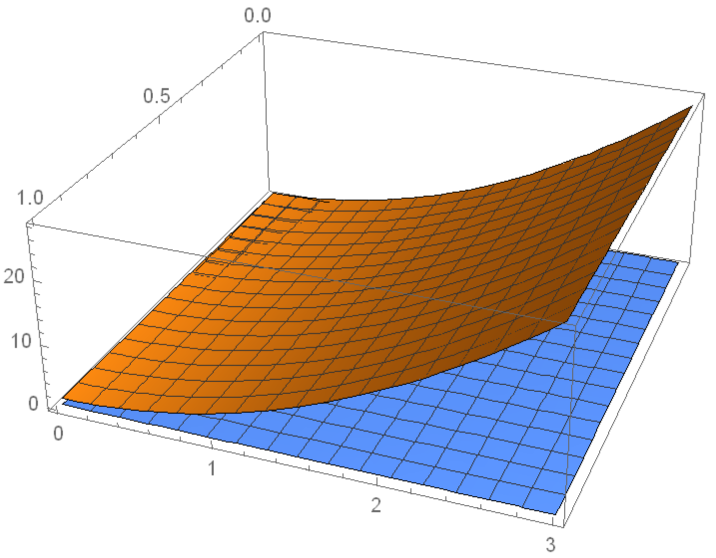
Visual: https://www.math3d.org/YKdmMST9

15.2 Double Integrals over General Regions

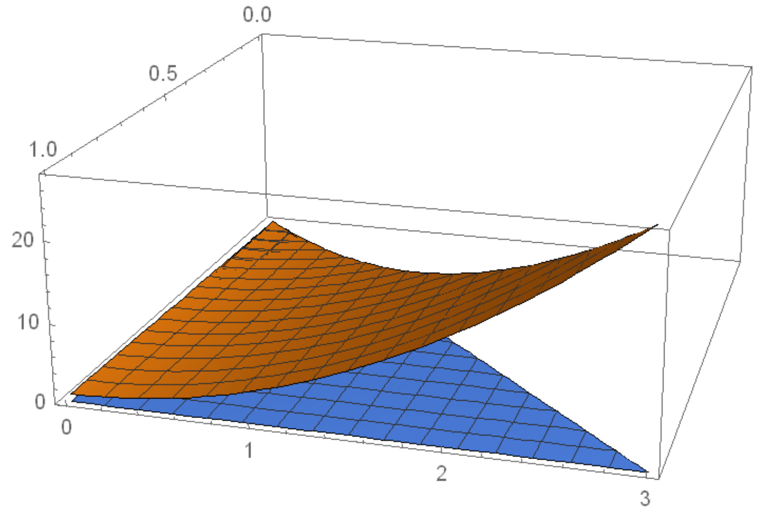
In 15.2, we discuss regions, *R*, other than rectangles.

Type 1 Regions (Top/Bot)	Type 2 Regions (Left/Right)
Given *x* in the range, a ≤ x ≤ b, we have $g_1(x) \le y \le g_2(x)$	Given *y* in the range, c ≤ y ≤ d, we have $h_1(y) \le x \le h_2(y)$
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$

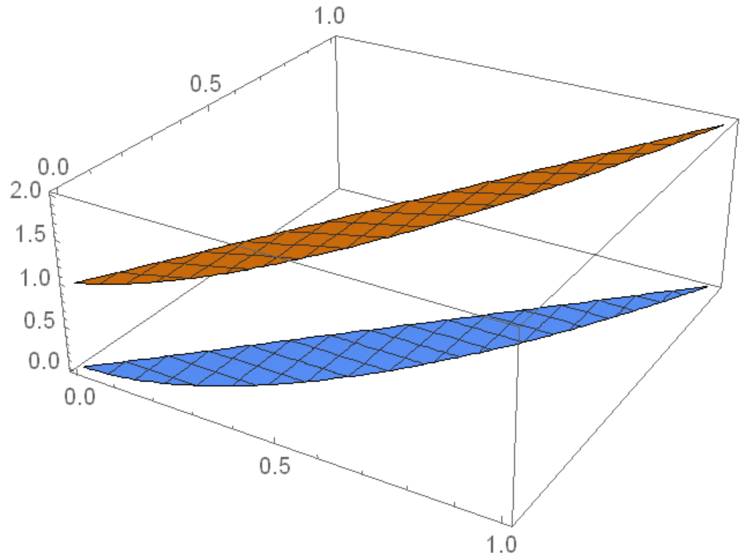
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



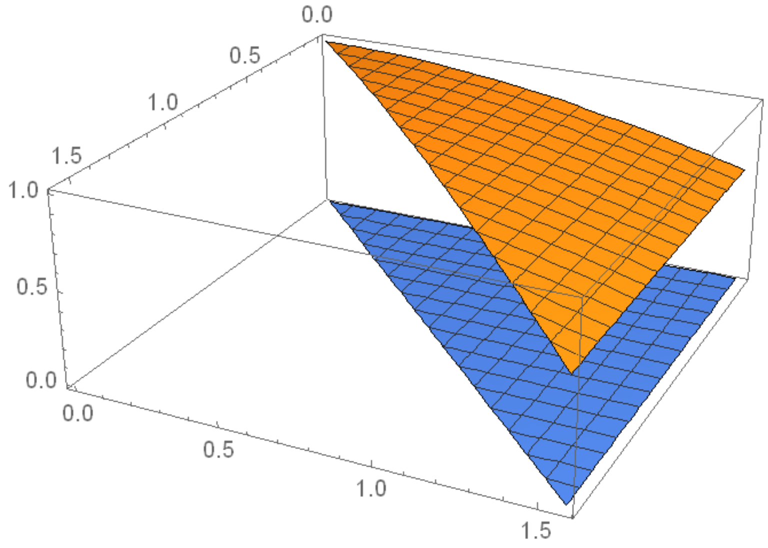
The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.



The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Example:

Draw the region, *R*, bounded by

$$y = x^2, y = 2x + 3.$$

Then set up the double integral

$$\iint_R f(x,y)dA$$

(Try it in both orders)

Visual: https://www.math3d.org/2fos5PZJ